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BOUNDS ON PHASE VELOCITY CHANGES OF PLANE WAVES IN ELASTIC MEDIA DUE TO CHANGES IN ELASTIC MODULI

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Abstract—By using the normality of the acoustic tensor for homogeneous, linearly elastic media, bounds are obtained for the changes in plane wave phase velocities due to changes in the elastic moduli of the medium. The bounds, which hold for arbitrary propagation directions and arbitrary changes in elastic moduli are then specialized to the cases where a single modulus is varied holding all the rest constant. Three dimensional bounding surfaces are derived and graphically presented for the possible changes in phase velocity (actually, square of phase velocity times mass density) of any given mode while propagating in any direction due to the changes in a specific elastic modulus. The results of this paper can be used in analytical developments wherever a need exists to place bounds on possible changes in a solution due to changes in elastic moduli, such as in the reconstruction of elastic moduli from phase or group velocity information. © 1997 Elsevier Science Ltd.

INTRODUCTION

It is known that the phase velocity of plane elastic waves in homogeneous, linearly elastic media are dependent upon the elastic moduli of the media, C_{ijkl}. A full discussion of the role of the elastic moduli in the elastodynamics of anisotropic media, as well as a discussion of some of the more important properties of this tensor can be found in Federov (1968).

It is an interesting problem, and perhaps useful in analytical developments, to be able to place bounds on the changes which can occur in the phase velocities of plane elastic waves due to changes in the elastic moduli, C_{ijkl}. Bounds for such changes could be useful when using a certain medium, with known elastic moduli, C^{ref}_{ijkl}, as a reference medium from which another medium, with moduli, $C_{ijkl} = C^{\text{ref}}_{ijkl} + \delta C_{ijkl}$ is established, or for approximating a material with a certain set of constants by another material having somewhat more symmetry. This is the case, for instance, in treating fibrous composites in which the fibers are aligned in a square array, by a transversely isotropic body. The actual elastic moduli will not, in general, satisfy the symmetry requirements of transverse isotropy; nevertheless, the approximation is used in many instances.

Being able to place analytical bounds on the changes which might occur in the phase velocities of the "perturbed" medium relative to the reference medium for propagation in any direction might help in bounding possible changes which could occur in solutions to problems in the reference medium when applied to problems in the perturbed medium. In addition, investigations of the possible changes in phase velocity due to the perturbation of individual elastic moduli show directly the effect which that constant can have on the propagation of plane elastic waves in the medium. Again with reference to composite materials, the bounds might be used to generate estimates for the errors which could arise from determining the elastic moduli from phase velocity measurements, in cases where the actual phase velocities arise from a slightly different set of elastic moduli than those assumed.

Substituting into the equations governing the motion of generally anisotropic elastic solids in the absence of body forces, viz,

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_j x_l} = \varrho\frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

the form of a plane wave,

$$u_i = A\hat{\xi}_i e^{ik(\hat{n}_m x_m - ct)} \tag{2}$$

leads directly to the requirement that

$$[C_{ijkl}\hat{n}_j\hat{n}_l - \varrho c^2\delta_{ik}]u_k = 0. \tag{3}$$

In eqns (2)–(3), $(\hat{\mathbf{n}}, \hat{\boldsymbol{\xi}}) \in \mathbb{R}^3 \times \mathbb{R}^3$ represent unit vectors in the directions perpendicular to the wavefronts and parallel to the particle vibration direction, respectively.

Defining the acoustic tensor, $\mathbf{Q}(\hat{\mathbf{n}})$, by the componentwise relation,

$$Q_{ik}(\hat{\mathbf{n}}) := C_{ijkl}\hat{n}_j\hat{n}_l, \tag{4}$$

eqn (3) can be recognized as a standard eigenvalue equation for $\mathbf{Q}(\hat{\mathbf{n}})$. the condition for existence of nontrivial solutions is,

$$\det[\mathbf{Q}(\hat{\mathbf{n}}) - \varrho c^2\mathbf{I}] = 0, \tag{5}$$

where $\mathbf{I}$ denotes the 3 by 3 identity matrix. It can be concluded from eqn (5) that the elastic moduli of a medium affect the phase velocities of plane waves in the medium only through the acoustic tensor. That is, changes in the elastic moduli which leave the components of the acoustic tensor unchanged also leave the phase velocities in the medium unchanged. It should be noted that since the acoustic tensor elements depend upon the unit vector $\hat{\mathbf{n}}$ the phase velocities, c, and wavenumber k, will also.

Changes in the elastic moduli will, in general, cause changes to the elements of the acoustic tensor, and also, therefore, in the phase velocities, c_α, $\alpha \in \{1, 2, 3\}$, for any propagation direction, $\hat{\mathbf{n}}$. As far as the magnitude of the phase velocity changes which can be expected due to changes in the moduli, it is shown in the appendix that the following upper bound is valid;

$$\sum_{\alpha=1}^{3} |\varrho c_\alpha^2(\hat{\mathbf{n}}) - \tilde{\varrho}\tilde{c}_\alpha^2(\hat{\mathbf{n}})|^2 \leqslant \|\mathbf{Q}(\hat{\mathbf{n}}) - \tilde{\mathbf{Q}}(\hat{\mathbf{n}})\|^2. \tag{6}$$

In eqn (6), c_α, $(\tilde{c}_\alpha)$ $\alpha \in \{1, 2, 3\}$ represent the 3 phase velocities of plane waves propagating in the direction $\hat{\mathbf{n}}$ in a medium with elastic moduli C_{ijkl} $(\tilde{C}_{ijkl})$ with the corresponding acoustic tensor $\mathbf{Q}(\hat{\mathbf{n}})$ $(\tilde{\mathbf{Q}}(\hat{\mathbf{n}}))$. $\|\mathbf{X}\|$ denotes the Euclidean norm of $\mathbf{X}$, as defined by eqn (A4) of the appendix.

Equation (6) is the desired bound on the possible changes in phase velocities of plane waves propagating in any direction, due to arbitrary changes in the elastic moduli. As expected, the possible changes are directly related to the changes which occur in the acoustic tensor components. The inequality in (6) is, for certain propagation directions and material symmetry groups, as good a bound as one can get since *equality* of the left and right hand sides is actually achieved.

As a simple example, consider two orthotropic media which differ only in the value of their C_{11} elastic modulus, i.e., $\tilde{C}_{11} \neq C_{11}$, but $\tilde{C}_{ij} = C_{ij} \forall (i, j) \neq (1, 1)$ and $\tilde{\varrho} = \varrho$. The axes to which the elastic moduli are referred (x_1, x_2, x_3) are assumed to be aligned with the planes of material symmetry of the orthotropic media. Under these circumstances, it can be shown (see eqn (11)) that the right hand side of eqn (6) reduces to the single term $\|\mathbf{Q}(\hat{\mathbf{n}}) - \tilde{\mathbf{Q}}(\hat{\mathbf{n}})\|^2 = |C_{11} - \tilde{C}_{11}|^2 l^4$, where l represents the direction cosine between the propagation direction, $\hat{\mathbf{n}}$, and the x_1 axis. For propagation along the x_1 axis, $l = 1$, and the right hand side therefore reduces to $|C_{11} - \tilde{C}_{11}|^2$. It is shown, for instance, in Auld (1992), that for propagation along the x_1 axis in an orthotropic medium, the three permissible plane wave phase velocities are given by $\varrho c_L^2 = C_{11}$, $\varrho c_{T1}^2 = C_{66}$ and $\varrho c_{T2}^2 = C_{55}$, where c_L represents the (pure) longitudinal wave phase velocity, and c_{T1}, c_{T2} represent the two pure shear wave velocities permissible in the $l = 1$ direction. Since only C_{11} differs between the two media, only the value of the longitudinal wave phase velocity will change, and hence, the left hand side of the inequality (6) also reduces to the single term, $|\varrho c_L^2 - \varrho \tilde{c}_L^2|^2 = |C_{11} - \tilde{C}_{11}|^2$, and hence eqn (6) reduces to an equality.

The inequality in (6) can be "dulled" somewhat by observing that the left hand side is the sum of three positive (or zero) terms. The fact that the sum of the three terms is less than the right hand side implies that each term is less than the right hand side, and one can write for *any* wave mode α, the further inequality,

$$|\varrho c_\alpha^2(\hat{\mathbf{n}}) - \tilde{\varrho}\tilde{c}_\alpha^2(\hat{\mathbf{n}})| \leqslant \| \mathbf{Q}(\hat{\mathbf{n}}) - \tilde{\mathbf{Q}}(\hat{\mathbf{n}}) \| \quad \alpha \in \{1, 2, 3\} \tag{7}$$

where the ordering, $c_1 \leqslant c_2 \leqslant c_3 \leqslant$, $\tilde{c}_1 \leqslant \tilde{c}_2 \leqslant \tilde{c}_3$ should be borne in mind.

In the next section these results are applied to the case where a single elastic constant is changed in going from "old" to "new" media. The resulting bounding surfaces offer a quick visual overview of the effects of individual elastic constants on the phase velocities of plane waves propagating in various directions in anisotropic elastic media.

The inequality in eqn (6) can be expanded fully once a coordinate basis is chosen for representation of the acoustic tensor and unit vector $\hat{\mathbf{n}}$. Assuming a Cartesian coordinate system is chosen, the Voight reduced index notation is used to refer to the elastic constants of the two media relative to the chosen coordinate basis. C_{11}, for example, refers to C_{1111}, etc. With respect to the chosen coordinate basis, let the unit vector $\hat{\mathbf{n}}$ have components, (l, m, n) with $l^2 + m^2 + n^2 = 1$. The quantities l, m and n therefore represent the cosines of the angles that the unit vector $\hat{\mathbf{n}}$ makes with the x_1, x_2 and x_3 coordinate axes, respectively.

It is then straightforward to show that the Euclidean norm of the difference between the acoustic tensors of two media with elastic moduli C_{ijkl} and $\tilde{C}_{ijkl}$, can be written,

$$\| \mathbf{Q} - \tilde{\mathbf{Q}} \| = \{\Delta Q_{11}^2 + \Delta Q_{22}^2 + \Delta Q_{33}^2 + 2\Delta Q_{12}^2 + 2\Delta Q_{13}^2 + 2\Delta Q_{23}^2\}^{1/2}, \tag{8}$$

where

$$\begin{aligned}
\Delta Q_{11} &= \Delta C_{11} l^2 + \Delta C_{66} m^2 + \Delta C_{55} n^2 + 2\Delta C_{16} lm + 2\Delta C_{56} mn + 2\Delta C_{15} ln \\
\Delta Q_{22} &= \Delta C_{66} l^2 + \Delta C_{22} m^2 + \Delta C_{44} n^2 + 2\Delta C_{26} lm + 2\Delta C_{24} mn + 2\Delta C_{46} ln \\
\Delta Q_{33} &= \Delta C_{55} l^2 + \Delta C_{44} m^2 + \Delta C_{33} n^2 + 2\Delta C_{45} lm + 2\Delta C_{34} mn + 2\Delta C_{35} ln \\
\Delta Q_{12} &= \Delta C_{16} l^2 + \Delta C_{26} m^2 + \Delta C_{45} n^2 + \Delta(C_{12} + C_{66}) lm + \Delta(C_{25} + C_{46}) mn + \Delta(C_{14} + C_{56}) ln \\
\Delta Q_{13} &= \Delta C_{15} l^2 + \Delta C_{46} m^2 + \Delta C_{35} n^2 + \Delta(C_{14} + C_{56}) lm + \Delta(C_{36} + C_{45}) mn + \Delta(C_{13} + C_{55}) ln \\
\Delta Q_{23} &= \Delta C_{56} l^2 + \Delta C_{24} m^2 + \Delta C_{34} n^2 + \Delta(C_{25} + C_{46}) lm + \Delta(C_{23} + C_{44}) mn + \Delta(C_{36} + C_{45}) ln
\end{aligned} \tag{9}$$

and,

$$\Delta C_{ij} := C_{ij} - \tilde{C}_{ij} \quad i, j \in \{1, 2, \ldots, 6\}. \tag{10}$$

As an example of the application of eqn (6), consider the case where only the elastic constant C_{11} is changed, all others being held constant. In this case, $\Delta C_{11} = C_{11} - \tilde{C}_{11} \neq 0$, but $\Delta C_{ij} = 0, (i, j) \neq (1, 1)$. Reference to eqn (9) then shows that $\| \Delta \mathbf{Q} \| := \| \mathbf{Q} - \tilde{\mathbf{Q}} \|$, reduces to the single term,

$$\begin{aligned}
\| \Delta \mathbf{Q} \| &= \{\Delta Q_{11}^2\}^{1/2} \\
&= \{\Delta C_{11}^2 l^4\}^{1/2} \\
&= |\Delta C_{11}| l^2
\end{aligned} \tag{11}$$

and therefore, eqn (7) reduces to,

$$|\varrho c_\alpha^2(\hat{\mathbf{n}})-\tilde{\varrho}\tilde{c}_\alpha^2(\hat{\mathbf{n}})| \leqslant |\Delta C_{11}|\, l^2 \quad \alpha\in\{1,2,3\}. \tag{12}$$

Inequality (12) states that there exists, in any two media in which only C_{11} differ, corresponding wave modes whose squared phase velocity (times mass density) differ by no more than the quantity $|\Delta C_{11}|\, l^2$. The "corresponding" modes could conceivably be different modes in the two media, i.e., they need not both be quasi-longitudinal or both quasi-transverse.

One can introduce the spherical polar coordinates (R,θ,ϕ) in place of (x_1,x_2,x_3) (where R is the distance from the origin, θ is the polar angle measured in the x_1-x_2 plane from the x_1 axis and ϕ is the "cone angle" measured from the x_3 axis) with the usual relations, $x_1 = R\sin(\phi)\cos(\theta)$, $x_2 = R\sin(\phi)\sin(\theta)$, $x_3 = R\cos(\phi)$ and the restrictions $R \geqslant 0$, $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \theta < 2\pi$. Using the spherical coordinates, the direction cosines, l, m and n can be written, $l = \sin(\phi)\cos(\theta)$, $m = \sin(\phi)\sin(\theta)$, and $n = \cos(\phi)$ with, of course, $l^2+m^2+n^2 = 1$.

In terms of these spherical coordinates, eqn (12) becomes,

$$|\varrho c_\alpha(\hat{\mathbf{n}})^2-\tilde{\varrho}\tilde{c}_\alpha(\hat{\mathbf{n}})^2| \leqslant |\Delta C_{11}|\sin^2(\phi)\cos^2(\theta) \quad \alpha\in\{1,2,3\}. \tag{13}$$

Figure 1a is a plot of the function on the right hand side of eqn (13) for $R = 1$, and $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \theta < 2\pi$. The plot shows graphically the (upper) bounding surface for the change in density times squared phase velocity of any plane wave, propagating in any direction in a generally anisotropic medium due to changes in C_{11} alone. It can be seen from Fig. 1a that changes in C_{11} can affect most severely the phase velocity of plane waves propagating in the direction of the x_1 axis. The maximum value of this change can be seen from eqn (12) or (13) to be $|\Delta C_{11}|/\varrho$ (assuming $\tilde{\varrho} = \varrho$). In addition, it is seen from the figure that changes in the elastic constant C_{11} will have no effect on the phase velocity of any of the three wave modes propagating in the plane $x_1 = 0$, i.e., the x_2-x_3 plane since the surface contracts to the origin in that plane.

Shown in Figs 1b and 1c are the bounding surfaces corresponding to elastic moduli C_{22} and C_{33} respectively. A derivation of the analytical form of these (and other) bounding surfaces is given below. As can be seen, these surfaces are identical in shape to that corresponding to changes in C_{11}, but their spatial orientation is different. Thus, C_{22} (C_{33}) has a most pronounced effect on waves propagating in the $\pm x_2$ ($\pm x_3$) direction, and no influence at all on any mode with wavevector lying entirely in the x_1-x_3 (x_1-x_2) plane. It will be seen below that certain other groups of moduli have bounding surfaces which are similar to each other, differing only in orientation.

The fact that certain moduli have no effect on waves whose wavevectors lie entirely in specific planes (or in specific isolated directions) could also have been arrived at by direct examination of the acoustic tensor components for specific choices of l, m and n. For example, since C_{11} only appears in the acoustic tensor through its product with l^2, it follows that if $l = 0$, the acoustic tensor, and hence all phase velocities, are independent of C_{11}. Direct examination of the acoustic tensor does not, however, give any estimate of the magnitude of possible changes in phase velocity associated with changes in C_{11} or any other modulus.

One can perform the same type of analysis as done for C_{11} with any of the 21 elastic constants of the medium. That is, eqn (7) can be used to place upper bounds on the changes in the product of density times squared phase velocity for changes in any of the elastic constants of the medium, holding all others constant. If this is done, it will be found that the bounding surface corresponding to changes in, say, elastic constant C_{ij} holding all others constant, can be written in the form,

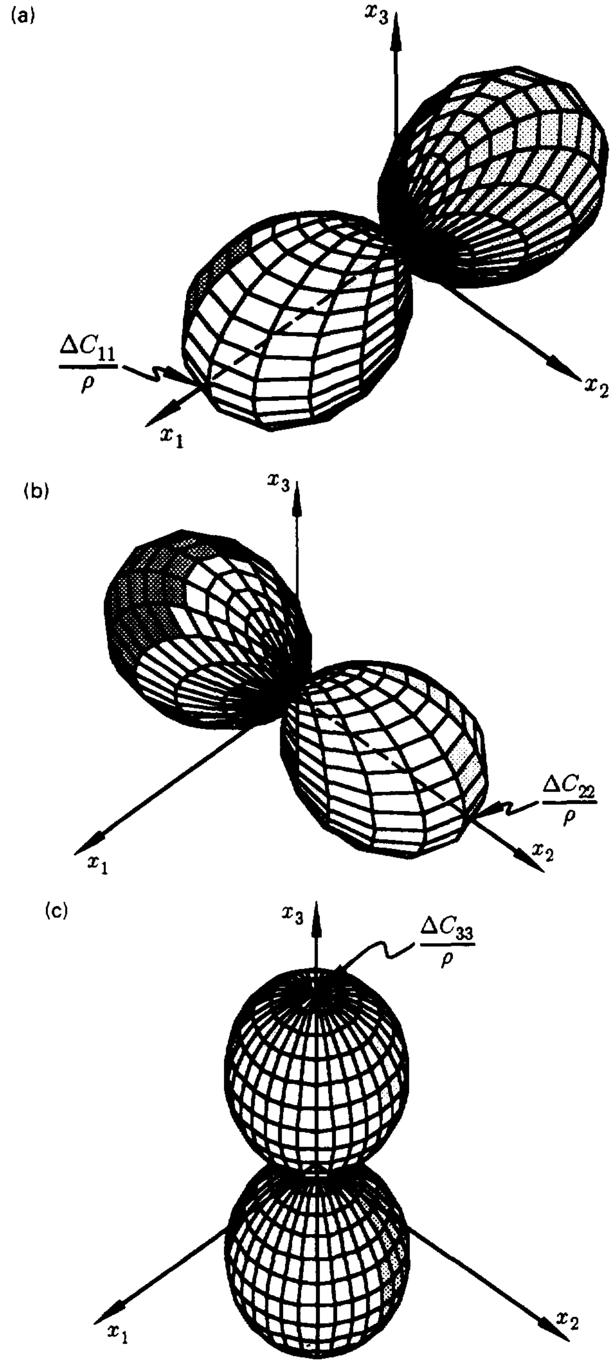

Fig. 1. Spherical polar plot of the three group I influence functions listed in Table 1. The surface represents the upper bound on changes which can occur in the phase velocity of any mode when propagating in any direction due to changes in a single modulus C_{ii}, $i \in \{1, 2, 3,\}$ alone. (a) $\Psi_{11}(\theta, \phi)$ corresponding to changes in the elastic constant C_{11}, (b) $\Psi_{22}(\theta, \phi)$ corresponding to changes in the elastic constant C_{22}, (c) $\Psi_{33}(\theta, \phi)$ corresponding to changes in the elastic constant C_{33}.

$$|\varrho c_\alpha^2(\hat{\mathbf{n}}) - \tilde{\varrho}\tilde{c}_\alpha^2(\hat{\mathbf{n}})| \leqslant |\Delta C_{ij}|\,\psi_{ij}(l, m, n) = |\Delta C_{ij}|\,\Psi_{ij}(\theta, \phi) \quad \alpha \in \{1, 2, 3\} \tag{14}$$

where the 21 "influence" functions, $\psi_{ij}(l, m, n)$ are listed in Table 1. Note that since $l^2 + m^2 + n^2 = 1$, only two of the three direction cosines are independent. The functional

Table 1. Influence functions corresponding to the elastic constants and their extremal properties

Group	Elastic constant	$\psi_{ij}(l,m,n)$	Maximum value	Direction(s) of principal maxima	Direction(s) of zeroes
I	ΔC_{11}	l^2	1	$l=\pm 1$	$l=0$
	ΔC_{22}	m^2	1	$m=\pm 1$	$m=0$
	ΔC_{33}	n^2	1	$n=\pm 1$	$n=0$
II	ΔC_{44}	n^2+m^2	1	$l=0$	$l=\pm 1$
	ΔC_{55}	l^2+n^2	1	$m=0$	$m=\pm 1$
	ΔC_{66}	l^2+m^2	1	$n=0$	$n=\pm 1$
III	ΔC_{12}	$\sqrt{2}\,\lvert lm\rvert$	$1/\sqrt{2}$	$n=0, l=\pm 1/\sqrt{2}$ $m=\pm 1/\sqrt{2}$	$l=0; m=0$
	ΔC_{13}	$\sqrt{2}\,\lvert ln\rvert$	$1/\sqrt{2}$	$m=0, l=\pm 1/\sqrt{2}$ $n=\pm 1/\sqrt{2}$	$l=0; n=0$
	ΔC_{23}	$\sqrt{2}\,\lvert mn\rvert$	$1/\sqrt{2}$	$l=0, m=\pm 1/\sqrt{2}$ $n=\pm 1/\sqrt{2}$	$m=0; n=0$
IV	ΔC_{45}	$\sqrt{2}\{n^2+2l^2m^2\}^{1/2}$	$\sqrt{2}$	$l=\pm 1/\sqrt{2}, m=\pm 1/\sqrt{2};$ $n=1$	$l=\pm 1; m=\pm 1$
	ΔC_{46}	$\sqrt{2}\{m^2+2l^2n^2\}^{1/2}$	$\sqrt{2}$	$l=\pm 1/\sqrt{2}, n=\pm 1/\sqrt{2};$ $m=1$	$l=\pm 1; n=\pm 1$
	ΔC_{56}	$\sqrt{2}\{l^2+2m^2n^2\}^{1/2}$	$\sqrt{2}$	$m=\pm 1/\sqrt{2}, n=\pm 1/\sqrt{2};$ $l=1$	$m=\pm 1; n=\pm 1$
V	ΔC_{14}	$\sqrt{2}\,\lvert l\rvert\{m^2+n^2\}^{1/2}$	$1/\sqrt{2}$	$l=\pm 1/\sqrt{2}$ $m^2+n^2=1/2$	$l=0; l=\pm 1$
	ΔC_{25}	$\sqrt{2}\,\lvert m\rvert\{l^2+n^2\}^{1/2}$	$1/\sqrt{2}$	$m=\pm 1/\sqrt{2}$ $l^2+n^2=1/2$	$m=0; m=\pm 1$
	ΔC_{36}	$\sqrt{2}\,\lvert n\rvert\{l^2+m^2\}^{1/2}$	$1/\sqrt{2}$	$n=\pm 1/\sqrt{2}$ $l^2+m^2=1/2$	$n=0; n=\pm 1$
VI	ΔC_{15}	$\sqrt{2}\,\lvert l\rvert\{l^2+2n^2\}^{1/2}$	$\sqrt{2}$	$l=\pm 1$	$l=0$
	ΔC_{16}	$\sqrt{2}\,\lvert l\rvert\{l^2+2m^2\}^{1/2}$	$\sqrt{2}$	$l=\pm 1$	$l=0$
	ΔC_{24}	$\sqrt{2}\,\lvert m\rvert\{m^2+2n^2\}^{1/2}$	$\sqrt{2}$	$m=\pm 1$	$m=0$
	ΔC_{26}	$\sqrt{2}\,\lvert m\rvert\{m^2+2l^2\}^{1/2}$	$\sqrt{2}$	$m=\pm 1$	$m=0$
	ΔC_{34}	$\sqrt{2}\,\lvert n\rvert\{n^2+2m^2\}^{1/2}$	$\sqrt{2}$	$n=\pm 1$	$n=0$
	ΔC_{35}	$\sqrt{2}\,\lvert n\rvert\{n^2+2l^2\}^{1/2}$	$\sqrt{2}$	$n=\pm 1$	$n=0$

forms $\Psi_{ij}(\theta,\phi)$ are obtained from $\psi_{ij}(l,m,n)$ by using spherical coordinates (R,θ,ϕ) in place of (l,m,n).

As can be seen from Table 1, the 21 elastic constants naturally separate into six groups as far as the influence functions defined in eqn (14) are concerned. This grouping of the constants is based on the fact that the influence function corresponding to any modulus in a particular group can be transformed into the influence function corresponding to any other modulus in the same group by a rotation or re-definition of the coordinate axes. Any elastic modulus within a single group therefore has fundamentally the same influence on the phase velocities of plane waves as any other; the only difference being the orientation of the bounding surface. The effects of the constants in different groups on the phase velocities in various directions can be rather different. Figures 2–6 show a representative sample of the bounding surfaces (i.e., spherical polar plots of $\Psi_{ij}(\theta,\phi)$) for one constant from each group from II through VI. The surfaces for other constants in each group can be obtained by rotations of the given plots about the coordinate axes, as shown in Fig. 1 for the moduli comprising group I.

DISCUSSION AND CONCLUSION

The "influence" functions, $\psi_{ij}(l,m,n)$ contain information concerning the directions for which the particular elastic constant, C_{ij} can influence the phase velocity of the plane

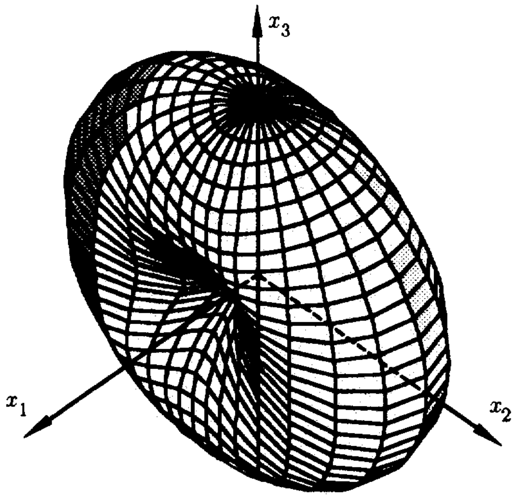

Fig. 2. Spherical polar plot of the influence function, $\Psi_{44}(\theta, \phi)$, corresponding to changes in the elastic constant C_{44} (see also Fig. 1).

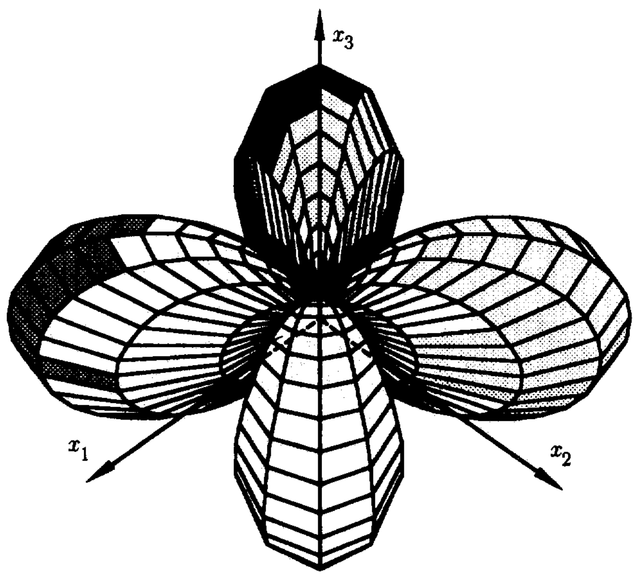

Fig. 3. Spherical polar plot of the influence function, $\Psi_{12}(\theta, \phi)$, corresponding to changes in the elastic constant C_{12} (see also Fig. 1).

elastic waves. It follows from eqn (14) that the phase velocity of all three modes are independent of the elastic constant, C_{ij}, for any direction in which the corresponding function ψ_{ij} vanishes. While this result follows from the present analysis in a straightforward manner, it should be noted that the same conclusions can be drawn by direct examination of the elements of the acoustic tensor relative to the chosen basis.

More importantly, perhaps, is the fact that the maximum value of the influence functions place upper bounds on the difference between the quantities ϱc_α^2 and $\tilde{\varrho}\tilde{c}_\alpha^2$ for any mode $\alpha \in \{1, 2, 3\}$ in the "original" medium and its corresponding mode in the perturbed medium. As a final note, the inequality in eqn (14) is once again generalized and simultaneously "dulled" into the nevertheless interesting corollary,

Corollary 1 : for arbitrary changes in the elastic constant C_{ij} of a medium, and any plane wave in the original medium, there exists a plane wave mode in the new medium whose squared phase velocity times mass density does not differ from that of the original medium by more than $\sqrt{2}\,|\Delta C_{ij}|$.

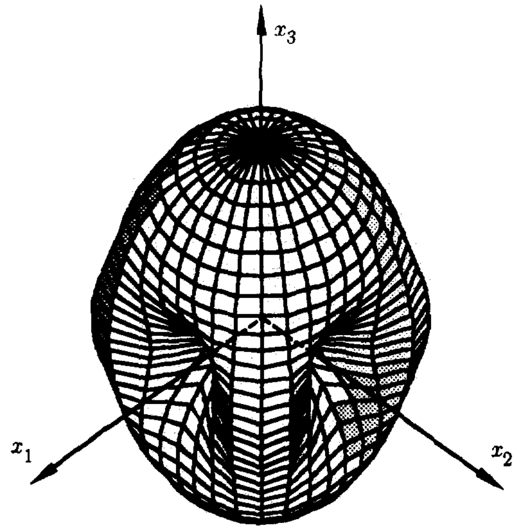

Fig. 4. Spherical polar plot of the influence function, Ψ_{45} (θ, ϕ), corresponding to changes in the elastic constant C_{45} (see also Fig. 1).

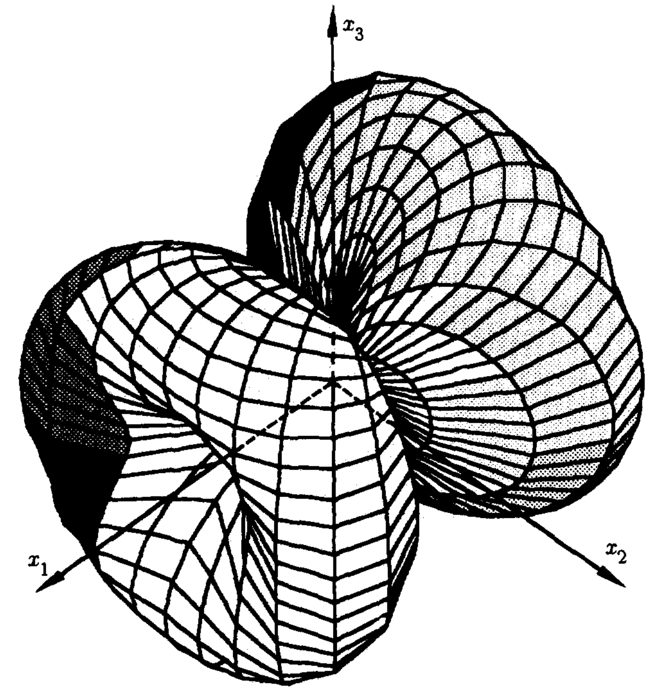

Fig. 5. Spherical polar plot of the influence function, Ψ_{14} (θ, ϕ), corresponding to changes in the elastic constant C_{14} (see also Fig. 1).

This result, which holds uniformly for any propagation direction, follows directly from the fact that the maximum value of any of the influence functions listed in Table 1 is $\sqrt{2}$.

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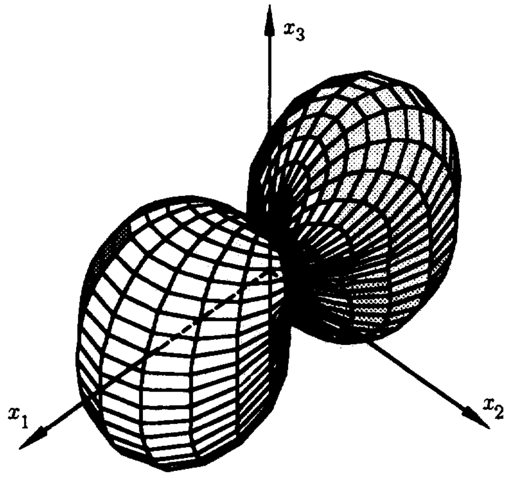

Fig. 6. Spherical polar plot of the influence function, $\Psi_{15}(\theta, \phi)$, corresponding to changes in the elastic constant C_{15} (see also Fig. 1).

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APPENDIX

In this appendix, use is made of the *normality* of the acoustic tensor to enable application of a theorem of matrix/functional analysis which furnishes the required bounds in terms of the *Euclidean norm* of the difference of the acoustic tensors of the two media. The results are valid for arbitrary changes in the elastic moduli, not just small changes. In what follows, the space of square, $n \times n$ matrices over the field of reals ($\mathbb{R}$) is denoted $\mathbf{M}_n(\mathbb{R})$, and σ denotes a particular permutation of the set $\{1, 2, 3\}$. The notation $\sigma(i)$ refers to the ith element in the permutation.

Recall that a matrix, $\mathbf{B} \in \mathbf{M}_n(\mathbb{R})$ is called *normal* (or sometimes *real normal*), if it satisfies the relation

$$\mathbf{B}\mathbf{B}^T = \mathbf{B}^T\mathbf{B}, \tag{A1}$$

(see Marcus and Minc (1992)) where $\mathbf{B}^T$ denotes the transpose of $\mathbf{B}$. Because of the symmetry of the acoustic tensor, $\mathbf{Q}(\hat{\mathbf{n}}) = \mathbf{Q}(\hat{\mathbf{n}})^T$ for any direction $\hat{\mathbf{n}}$ (see Federov (1968)), it follows that $\mathbf{Q}(\hat{\mathbf{n}})$ is normal for any $\hat{\mathbf{n}} \in \mathbb{R}^3$.

Denoting the characteristic values of the acoustic tensor for any given direction $\hat{\mathbf{n}}$ by $\lambda_1^Q, \lambda_2^Q, \lambda_3^Q$, eqn (5) shows that the following relations hold,

$$\lambda_\alpha^Q = \varrho c_\alpha^2(\hat{\mathbf{n}}) \quad \alpha \in \{1, 2, 3\}. \tag{A2}$$

It is noted that $\lambda_\alpha^Q \in \mathbb{R}^+$ (i.e., real and strictly positive) for $\alpha \in \{1, 2, 3\}$ because the acoustic tensor is both Hermitian and positive definite. Because of this, the characteristic values of the acoustic tensor can be ordered, and the following ordering will be assumed from here on,

$$\lambda_1^Q \leqslant \lambda_2^Q \leqslant \lambda_3^Q. \tag{A3}$$

Finally, the Euclidean norm of the matrix $\mathbf{B} \in \mathbf{M}_n(\mathbb{R})$, denoted by $\| \mathbf{B} \|$, is defined by

$$\| \mathbf{B} \| := \mathrm{tr}(\mathbf{B}\mathbf{B}^T)^{1/2} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}^2 \right)^{1/2}. \tag{A4}$$

To obtain the desired bounds on the changes of wave speed associated with changes of elastic moduli, use is made of the following theorem of matrix analysis proven, for instance, in Hoffman and Wielandt (1953), and Marcus and Minc (1992):

Theorem 1: if $\mathbf{A} \in \mathbf{M}_n(\mathbb{R})$ and $\mathbf{B} \in \mathbf{M}_n(\mathbb{R})$ are normal with characteristic roots $\lambda_1^A, \lambda_2^A, \ldots, \lambda_n^A$ and $\lambda_1^B, \lambda_2^B, \ldots, \lambda_n^B$, respectively, then there exists an ordering $\lambda_{\sigma(1)}^B, \lambda_{\sigma(2)}^B, \ldots, \lambda_{\sigma(n)}^B$ such that,

$$\sum_{i=1}^{n} | \lambda_i^A - \lambda_{\sigma(i)}^B |^2 \leqslant \| \mathbf{A} - \mathbf{B} \|^2. \tag{A5}$$

where $\| \mathbf{X} \|$ denotes the Euclidean norm of $\mathbf{X}$.

Theorem 1, when applied to the acoustic tensors of media with differing elastic moduli will yield the desired bounds on possible phase velocity changes. Before applying the theorem, however, use is made of the fact that the acoustic tensor for linearly elastic media is not only normal, but Hermitian; hence its characteristic values are real, and can be ordered according to Eqn (A3). This particular ordering for the characteristic values of **A** and **B** will then minimize the left hand side of eqn (A5), and hence it suffices to let $\sigma = \{1, 2, 3\}$, i.e., $\sigma(i) = i$ in theorem 1.

Proceeding, we associate with the matrices **A** and **B** in theorem 1 the acoustic tensors of two media with elastic moduli C_{ijkl} and $\tilde{C}_{ijkl}$ and mass densities, ϱ and $\tilde{\varrho}$ which are denoted $\mathbf{Q}(\hat{\mathbf{n}})$ and $\tilde{\mathbf{Q}}(\hat{\mathbf{n}})$, respectively. Both acoustic tensors are in $\mathbf{M}_3(\mathbb{R})$ for any $\hat{\mathbf{n}} \in \mathbb{R}^3$. The eigenvalues of $\mathbf{Q}(\hat{\mathbf{n}})$ and $\tilde{\mathbf{Q}}(\hat{\mathbf{n}})$ are denoted $\lambda_\alpha(\hat{\mathbf{n}})$ and $\tilde{\lambda}_\alpha(\hat{\mathbf{n}})$, $\alpha \in \{1, 2, 3\}$, respectively. Denoting by $c_\alpha(\hat{\mathbf{n}})$ and $\tilde{c}_\alpha(\hat{\mathbf{n}})$, $\alpha \in \{1, 2, 3\}$, the phase velocities of the three plane waves permissible for wavevectors in the direction of $\hat{\mathbf{n}}$, eqn (A2) relates the characteristic values to the phase velocities.

A direct application of theorem 1 then yields,

Corollary 2: given two homogeneous linearly elastic media and an arbitrary unit vector $\hat{\mathbf{n}} \in \mathbb{R}^3$, denote the mass density, phase velocities and acoustic tensor of the first medium by ϱ, $c_1 \leqslant c_2 \leqslant c_3$ and $\mathbf{Q}(\hat{\mathbf{n}})$ respectively, and those of the second medium by $\tilde{\varrho}$, $\tilde{c}_1 \leqslant \tilde{c}_2 \leqslant \tilde{c}_3$ and $\tilde{\mathbf{Q}}(\hat{\mathbf{n}})$, respectively. Then,

$$\sum_{\alpha=1}^{3} |\varrho c_\alpha^2(\hat{\mathbf{n}}) - \tilde{\varrho}\tilde{c}_\alpha^2(\hat{\mathbf{n}})|^2 \leqslant \|\mathbf{Q}(\hat{\mathbf{n}}) - \tilde{\mathbf{Q}}(\hat{\mathbf{n}})\|^2, \tag{A6}$$

which is the desired bound on the changes in phase velocity due to changes in elastic moduli.